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UNCONDITIONALITY IN TENSOR PRODUCTS

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ABSTRACT
It is proved that in order to study unconditional structures in tensor products of finite dimensional Banach spaces it is enough to consider a certain basis. This result is applied to spaces of p-absolutely summing operators showing their "bad" structure.

0. Preliminaries

Most of our notations will coincide with the notations in [2] or [5]. The unconditional basis constant of a basis $\{x_i\}_{i=1}^n$ of a Banach space E with biorthogonal system $\{x_i^*\}_{i=1}^n$ is given by

$$\chi(\{x_i\}_{i=1}^n):=\sup\left\{\left\|\sum_{i=1}^n \varepsilon_i\langle x_i^*, x\rangle x_i\right\| \,\Big|\, \|x\|=1, \varepsilon_i=\pm 1\right\}$$

and the unconditional basis constant of a n-dimensional Banach space E by

$$\chi(E):=\inf\{\chi(\{x_i\}_{i=1}^n) \,|\, \{x_i\}_{i=1}^n \text{ is basis}\}.$$

We say that a Banach space G has locally unconditional structure (LUST) if there is a constant K such that for each finite dimensional subspace E there is another finite dimensional subspace $F\supset E$ with $\chi(F)\leqq K$.

Moreover, a Banach space G has GL-LUST [1] if there is a K such that for each finite dimensional subspace E there is a finite dimensional space F with $\chi(F)=1$ and operators $T\in L(E,F)$ and $S\in L(F,G)$ such that $ST_{|E}=\mathrm{id}$ and $\|S\|\,\|T\|\leqq K$. The infimum of all the numbers K is denoted by $\chi_u(G)$. A Banach space with LUST has GL-LUST.

As the main result we get that if a space has an unconditional basis, every basis which is "nearly" unconditional must already be unconditional. As an applica-

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tion we extend results of Gordon and Lewis [1] concerning spaces of p-absolutely summing operators $\Pi_p(E,F)$.

To get these results we use Walsh matrices defined by

$$W_1: = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad W_{n+1}: = \begin{pmatrix} W_n & W_n \\ W_n & -W_n \end{pmatrix}.$$

We will compute the p-absolutely summing norms of W_n mapping l^r_n onto l^s_n. In order to avoid confusion we sometimes denote the operator norm by $\| \ \|_{r,s}$.

Concerning elementary facts about tensor products we refer to [6]. For finite-dimensional spaces E, F we have $E \otimes_\varepsilon F = L(E^*, F)$ with the operator norm and $(E^* \otimes_\pi F^*)^* = E \otimes_\varepsilon F$.

The isomorphic distance of two Banach spaces E and F is defined by

$$\mathrm{d}(E,F): = \inf\{\|J\| \|J^{-1}\| \mid J \in L(E,F), J \text{ is isomorphism}\}.$$

If there is no isomorphism we set $\mathrm{d}(E,F): = \infty$. Furthermore we consider the norm $\gamma_p(A)$ of operators that factor through L_p-spaces. Since we are just concerned with finite-dimensional spaces E and F we have

$$\gamma_p(A) = \inf\{\|B\| \|C\| \mid A = B \cdot C, C \in L(E, l^p), B \in L(l^p, F)\}.$$

Gordon and Lewis [1] proved that for all finite dimensional Banach spaces E and F and for all operators $A \in L(E,F)$

$$\gamma_1(A) \leqq \chi_u(E)\pi_1(A), \tag{1}$$

where $\pi_1(\ )$ is the 1-absolutely summing norm.

1. An estimation of $\chi_u(E)$

The aim of this paragraph is to prove the following theorem. $|G|$ denotes the cardinality of a set G.

THEOREM 1. *Let $\{x_i\}_{i=1}^n$ be a basis of a Banach space E. Assume that there exist constants M and K and a set G of n-tuples of signs θ so that*

$$\Big\| \sum_{i=1}^n \theta_i a_i x_i \Big\| \leqq K \Big\| \sum_{i=1}^n a_i x_i \Big\| \quad \textit{for all scalars } \{a_i\}_{i=1}^n \textit{ and all } \theta = \{\theta_i\}_{i=1}^n \in G, \tag{2}$$

$$\frac{1}{|G|} \sum_{\theta \in G} \Big| \sum_{i=1}^n \theta_i \lambda_i \Big| \geqq \frac{1}{M} \Big(\sum_{i=1}^n |\lambda_i|^2 \Big)^1 \quad \textit{for all scalars } \{\lambda_i\}_{i=1}^n. \tag{3}$$

Then

$$\chi(\{x_i\}_{i=1}^n) \leqq K^2 M^2 \chi_u(E).$$

For the proof we need two propositions. We will consider diagonal mappings $T \in L(E^*, l_n^2)$ such that

$$T\left(\sum_{i=1}^{n} a_i x_i^*\right) = (t_i a_i)_{i=1}^{n}.$$

We want to estimate $\pi_1(T)$ and $\gamma_1(T)$. Similarly as in [1] we prove first

PROPOSITION 2. *Let $\{x_i\}_{i=1}^n$ be a basis of a Banach space E and suppose* (2) *and* (3) *are valid. Then we have for all diagonal operators $T \in L(E^*, l_n^2)$*

$$\pi_1(T) \leqq KM \left\| \sum_{i=1}^{n} t_i x_i \right\|,$$

PROOF. By (3) we have

$$\begin{aligned} \frac{1}{M} \sum_{l=1}^{N} \left(\sum_{i=1}^{n} |a_i^l t_i|^2 \right)^{\frac{1}{2}} &\leqq \frac{1}{|G|} \sum_{l=1}^{N} \sum_{\theta \in G} \left| \sum_{i=1}^{n} \theta_i a_i^l t_i \right| \\ &\leqq \max_{\theta \in G} \sum_{l=1}^{N} \left| \sum_{i=1}^{n} \theta_i a_i^l t_i \right| \\ &\leqq \max_{\theta \in G} \max_{\|x\|=1} \left\| \sum_{i=1}^{n} \theta_i t_i x_i \right\| \sum_{l=1}^{N} \left| \left\langle \sum_{i=1}^{n} a_i^l x_i^*, x \right\rangle \right| \\ &\leqq K \left\| \sum_{i=1}^{n} t_i x_i \right\| \max_{\|x\|=1} \sum_{l=1}^{N} \left| \left\langle \sum_{i=1}^{n} a_i^l x_i^*, x \right\rangle \right| \quad \square \end{aligned}$$

PROPOSITION 3. *Let $\{x_i\}_{i=1}^n$ be a basis of a Banach space E and suppose* (2) *and* (3) *are valid. Then we have for all diagonal operators $T \in L(E^*, l_n^2)$*

$$\max_{\pm} \left\| \sum_{i=1}^{n} \pm t_i x_i \right\| \leqq KM \gamma_1(T). \tag{4}$$

PROOF. We will prove (4) for

$$\max_{\|\Sigma_{i=1}^n a_i x_i^*\|=1} \sum_{i=1}^{n} |a_i t_i| = \max_{\pm} \left\| \sum_{i=1}^{n} \pm t_i x_i \right\|.$$

We have $\gamma_1(T) = \gamma_\infty(T')$ and $T' \in L(l_n^2, E)$. Moreover, we have

$$\gamma_\infty(T') = \inf\{\|B\| \|C\| \mid T' = C \cdot B, B \in L(l_n^2, l^\infty), C \in L(l^\infty, E)\}.$$

Obviously we can assume without restriction that B is a special embedding of l_n^2 into l^∞. Such an embedding is given by

$$J(x) := (\langle x, y_r \rangle)_{r \in \mathbf{N}},$$

where $\{y_r \mid r \in \mathbf{N}\}$ is a dense set in the unit sphere of l_n^2. By a standard approximation argument we can restrict ourselves to consider mappings $J \in L(l_n^2, l_N^\infty)$, $N \in \mathbf{N}$, with

$$J(x) := (\langle x, y_r \rangle)_{r=1}^N.$$

Thus, in order to compute $\gamma_\infty(T')$ we have to consider all mappings $R \in L(l_N^\infty, E)$ with $R \cdot J = T'$ and we have to estimate $\inf \| R \|$. As a representation for R we choose

$$R(x) = \sum_{i=1}^n \langle f_i, x \rangle t_i x_i$$

with

$$\delta_{ij} = \langle f_i, J(e_j) \rangle = \sum_{r=1}^N f_i(r) y_r(j), \tag{5}$$

where $f_i(r)$, $r = 1, \cdots, N$, denote the components of the biorthogonal functionals f_i and e_j, $j = 1, \cdots, n$, the natural unit vectors in l_n^2. Denoting $x^* = \sum_{k=1}^n a_k x_k^*$ with $\| x^* \| = 1$ we have

$$\begin{aligned} \| R \| &= \max_{\|z\|_\infty = 1} \left\| \sum_{i=1}^n \langle f_i, z \rangle t_i x_i \right\| \\ &= \max_{\|z\|_\infty = 1} \max_{\|x^*\|=1} \left| \sum_{i=1}^n \langle f_i, z \rangle \langle t_i x_i, x^* \rangle \right| \\ &= \max_{\|x^*\|=1} \max_{\|z\|_\infty = 1} \left| \sum_{i=1}^n \langle t_i a_i f_i, z \rangle \right| \\ &= \max_{\|x^*\|=1} \sum_{r=1}^N \left| \sum_{i=1}^n t_i a_i f_i(r) \right|. \end{aligned}$$

By (2) it follows

$$\geqq \max_{\|x^*\|=1} \frac{1}{|G|} \frac{1}{K} \sum_{\theta \in G} \sum_{r=1}^N \left| \sum_{i=1}^n \theta_i t_i a_i f_i(r) \right|.$$

By (3) we get

$$\| R \| \geqq \frac{1}{KM} \max_{\|x^*\|=1} \sum_{r=1}^N \left(\sum_{i=1}^n | t_i a_i f_i(r) |^2 \right)^{1/2}, \tag{6}$$

where $x^* = \sum_{i=1}^n a_i x_i^*$. On the other hand by using (5),

$$1 = \sum_{r=1}^N f_i(r) y_r(i) \qquad \text{for all } i = 1, \cdots, n,$$

we get

$$\sum_{i=1}^{n} |t_i a_i| = \sum_{r=1}^{N} \sum_{i=1}^{n} |t_i a_i| f_i(r) y_r(i)$$

and by the Hölder-inequality and $\|y_r\|_2 \leqq 1$ we have

$$\leqq \sum_{r=1}^{N} \left(\sum_{i=1}^{n} |t_i a_i f_i(r)|^2 \right)^{1/2}.$$

By this and (6) the proposition is proved. □

PROOF OF THEOREM 1. By a result (1) of Gordon and Lewis we have that

$$\gamma_1(T) \leqq \pi_1(T) \chi_u(E^*) \qquad \text{for all } T.$$

Observing $\chi_u(E^*) = \chi_u(E)$ and applying Propositions 2 and 3 we have proved Theorem 1. □

2. Unconditional matrix norms

We say a norm of the space of $n \times m$-matrices is an unconditional matrix norm if

$$\|(\varepsilon_i \eta_j a_{ij})_{i,j=1}^{n,m}\| = \|(a_{ij})_{i,j=1}^{n,m}\| \quad \text{for all } a_{ij} \in \mathbf{R}, \quad i = 1, \cdots, n,$$
$$j = 1, \cdots, m.$$

Especially, a tensor product norm on $E \otimes F$, where E and F are finite dimensional Banach spaces with unconditional bases, is an unconditional matrix norm. Let E_{ij} denote the matrix whose (i, j)-component is one and whose other components are zero.

COROLLARY 4. *Let F be the space of $n \times m$-matrices provided with an unconditional matrix norm. Then*

$$\chi(\{E_{ij}\}_{i,j=1}^{n,m}) \leqq 4\chi_u(F).$$

PROOF. As a basis in Theorem 1 we choose $\{E_{ij}\}_{i,j=1}^{n,m}$ and $G = \{(\varepsilon_i \eta_j)_{i,j=1}^{n,m} \mid \varepsilon_i = \pm 1,\ \eta_j = \pm 1\}$. Considering that F is provided with an unconditional matrix norm we get (2) with $K = 1$. By using the Khintchin-inequality [4] or [7] twice and the triangle-inequality once we have (3) with $M = 2$. □

We give now applications of Corollary 4 and extend some results obtained by Gordon and Lewis [1]. First we consider the ε- and π-tensor product.

LEMMA 5. *Suppose $\chi(E) = \chi(F) = 1$ and* $\dim E = n$, $\dim F = m$.

Then

$$\chi(E \otimes_\varepsilon F) \leqq \min\{d(E, l_n^\infty), d(F, l_m^\infty)\} \tag{7}$$

and

$$\chi(E \otimes_\pi F) \leqq \min\{d(E, l_n^1), d(F, l_m^1)\}. \tag{8}$$

PROOF. One gets (8) from (7) by dualization. Without restriction we are allowed to assume that

$$\min\{d(E, l_n^\infty), d(F, l_m^\infty)\} = d(F, l_m^\infty).$$

J denotes an isomorphism between F and l_m^∞. By this we have that

$$I : E \otimes_\varepsilon F \to E \otimes_\varepsilon l_m^\infty$$

with $I(A) := J \cdot A$ is an isomorphism with $\|I\| \|I^{-1}\| \leqq \|J\| \|J^{-1}\|$ and we get

$$d(E \otimes_\varepsilon F, E \otimes_\varepsilon l_m^\infty) \leqq d(F, l_m^\infty). \tag{9}$$

On the other hand we have because of $\chi(E) = 1$

$$\chi(E \otimes_\varepsilon l_m^\infty) = 1. \tag{10}$$

With (9) and (10) the proposition is proved. □

In order to get estimations for $\chi_u(l_n^p \otimes_\varepsilon l_n^r)$ from below it is according to Corollary 4 enough to estimate the unconditional basis constant of a certain basis. It turns out that we have to consider operators represented by Walsh matrices in order to get the estimations. We need the following lemma [3].

LEMMA 6. *Let W be a Walsh matrix of rank $n = 2^k$. For the operator norm of $W : l_n^p \to l_n^q$, $1 \leqq p, q \leqq \infty$, we have*

(i)	$\|W\| \leqq n^{1/q}$	$1 \leqq p, q \leqq 2$,
(ii)	$\|W\| \leqq \max\{n^{1/p'}, n^{1/q}\}$	$1 \leqq p \leqq 2 \leqq q \leqq \infty$,
(iii)	$\|W\| \leqq n^{1/2+1/q-1/p}$	$1 \leqq q \leqq 2 \leqq p \leqq \infty$,
(iv)	$\|W\| \leqq n^{1/p'}$	$2 \leqq p, q \leqq \infty$.

PROPOSITION 7.

(i) $1 \leqq r, s \leqq 2$

$$\frac{1}{8}\sqrt{n} \leqq \chi_u(l_n^r \otimes_\varepsilon l_n^s) \leqq (1 + \sqrt{2})\sqrt{n};$$

(ii) $1 \leqq r \leqq 2 \leqq s \leqq \infty$ *or* $2 \leqq r \leqq s \leqq \infty$

$$\frac{1}{8} n^{1/s} \leqq \chi_u(l_n^r \otimes_\varepsilon l_n^s) \leqq n^{1/s}.$$

PROOF. The estimations from above follow from Lemma 5 and a result of Gurarii, Kadec and Macaev [3]. The estimations from below are gained from Lemma 6 and Corollary 4: It is enough to consider quotients of $\|(1,\cdots,1)\otimes(1,\cdots,1)\| = n^{1/r+1/s}$ and $\|W\|$. Moreover, we can restrict ourselves to the case $n = 2^k$. □

Considering Proposition 7 one might state the following problem: Is E a $\mathscr{L}_\infty$-space if and only if $E \hat{\otimes}_\varepsilon E$ has LUST (or GL-LUST)?

Now we consider spaces of p-absolutely summing operators.

PROPOSITION 8. *Let*

$$\frac{1}{t} := \max\left\{\frac{1}{\max\{p,r,2\}} - \frac{1}{s}, \frac{1}{\max\{p',s,2\}} - \frac{1}{r}\right\}.$$

Then

$$\tfrac{1}{8} n^{1/t} \leqq \chi_u(\Pi_p(l_n^r, l_n^s)).$$

PROOF. Because of Corollary 4 it is enough to consider a certain basis. Moreover, we can restrict ourselves to the case $n = 2^k$ (admitting a factor $\frac{1}{2}$) and computing the quotient of $\pi_p((1,\cdots,1)\otimes(1,\cdots,1)) = n^{1/r'+1/s}$ and $\pi_p(W)$. Let us first treat the case $\max\{p,r,2\} < s$. The n column vectors of the Walsh matrix are denoted by w_i, $i = 1,\cdots,n$.

$$\pi_p(W) \geqq \inf\left\{ C \;\middle|\; \left(\sum_{i=1}^n \|W(w_i)\|_s^p\right)^{1/p} \leqq C \sup_{\|y\|_{r'}=1} \left(\sum_{i=1}^n |\langle w_i, y\rangle|^p\right)^{1/p}\right\}$$

$$= \inf\{C \mid n^{1+1/p} \leqq C\|W\|_{r',p}\}.$$

Applying Lemma 6 we get the first part of the estimation. We consider now the case $\max\{p',s,2\} < r$. By e_j, $j = 1,\cdots,n$ we denote the natural unit vectors.

$$\pi_p(W) = \inf\left\{C \;\middle|\; \sum_{i=1}^N \|W(x_i)\|_s^p \leqq C^p \sup_{\|y\|_r=1} \sum_{i=1}^N |\langle x_i, y\rangle|^p, x_i \in l_n^r\right\}$$

$$\leqq \inf\left\{C \;\middle|\; \sum_{i=1}^N \|W(x_i)\|_s^p \leqq C^p \sup_{j\leqq n} \sum_{i=1}^N |\langle x_i, e_j\rangle|^p, x_i \in l_n^r\right\}$$

$$\leqq \inf\left\{C \;\middle|\; \sum_{i=1}^N \|W(x_i)\|_s^p \leqq C^p \frac{1}{n} \sum_{i=1}^N \|x_i\|_p^p, x_i \in l_n^r\right\}$$

$$\leqq n^{1/p}\|W\|_{p,s}.$$

Now we apply Lemma 6. □

REMARK. It can be shown that $\chi_u(\Pi_p(l_n^r, l_n^s))$ is uniformly bounded for $1 \leqq p \leqq s \leqq 2 \leqq r \leqq p'$ [1].

COROLLARY 9. *Let E be a $\mathcal{L}_r$-space and F a $\mathcal{L}_s$-space, $1 \leqq r,\ s \leqq \infty$. Suppose that* $\max\{p, r, 2\} < s$ *or* $\max\{p', s, 2\} < r$. *Then* $\Pi_p(E, F)$ *does not have GL-LUST.*

PROOF. There are spaces $\Pi_p(l_n^r, l_n^s)$ uniformly complemented in $\Pi_p(E, F)$. By Proposition 8 and a result of Gordon and Lewis [1] the corollary follows. □

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REFERENCES

1. Y. Gordon and D. R. Lewis, *Absolutely summing operators and local unconditional structures*, Acta Math. **133** (1974), 24–48.

2. Y. Gordon, D. R. Lewis and J. R. Retherford, *Banach ideals of operators with applications*, J. Functional Analysis **14** (1973), 85–129.

3. V. I. Gurarii, M. E. Kadec and V. I. Macaev, *On the distance between isomorphic L_p-spaces of finite dimension*, Mat. Sb. **70** (112), 4 (1966), 481–489.

4. U. Haagerup, *Les meilleurs constantes de l'inégalité de Khintchine*, C. R. Acad. Sci. Paris, Ser. A, **286** (1978), 259–262.

5. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, 1977.

6. P. Saphar, *Produits tensoriels d'espaces de Banach et classes d'applications*, Studia Math. **28** (1970), 71–100.

7. S. J. Szarek, *On the best constants in the Khintchin-inequality*, Studia Math. **58** (1976), 197–208.

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